Some relations in the triangle

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1. On the sides of a triangle D construct equilateral triangles. The three centers of the equilateral triangles are either on opposite sides or on the same sides as the center of D and form an equilateral triangle D_1 or an equilateral triangle D_2. In Figure 1 we denote D, D_1, D_2 by ABC, $A_1B_1C_1$, $A_2B_2C_2$, respectively.

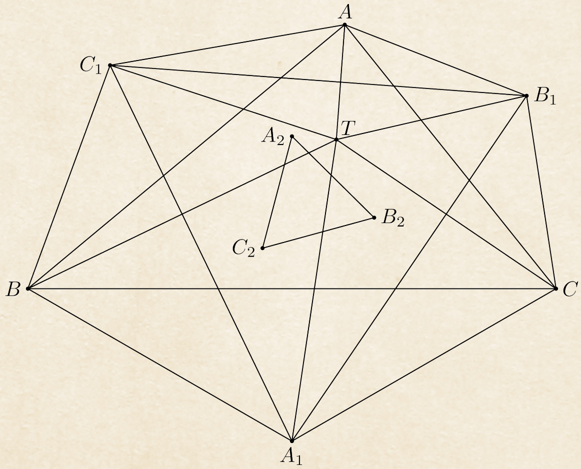

Figure 1.

The geometric proof of this fact, which one can get without great effort is not shown here. For the rest, reference can be made to a work by W. Fischer[1] and to the generalization considered in Section 4. The configuration with the equilateral triangles occurs under different names in the triangular geometry and is named after Torricelli[2]. In connection with the question of the equilateral triangle having the largest area we mention E.Fasbender[3]. The same configuration occurs in M. Filip[4], which helps determine the point minimizing the sum of distances to corners.

In this note, the above mentioned equilateral triangles D_1 and D_2 will be investigated, showing that they are connected to the original triangle by notable relationships. For the sake of clarity, the results in Section 2 underline this connection and proofs are given in Section 3. In Section 4, the relation (I) is extended to the general case of other attached triangles, whereby some applications result, and in Section 5 it is shown that the orthocenters of the triangles D_1 and D_2 coincide at a point that is the center of gravity of D when D_1 and D_2 are equilateral.

[1] Arch. Math. Phys. 40 (1863), p. 460

[2] Enzyklopadie der Math. Wiss. III AB 10, p. 1218

[3] Journ. fur Math. 30, p. 230—231 (1846)

[4] Gazeta mat. Bukarest 13, p. 68—71. compare with Fortschritte der Math Jahrgang 1907, p. 541

2. We denote the areas and the square perimeters (sum of the side squares) of the triangles D, D_1, D_2 with F, F_1, F_2 and S, S_1, S_2. The following relations hold in the case of equilateral attachment triangles:

$$F_1 - F_2 = F \tag{I}$$

$$S_1 + S_2 = S \tag{II}$$

$$S + 4\sqrt{3}F = 8\sqrt{3}F_1 \tag{III}$$

$$S - 4\sqrt{3}F = 8\sqrt{3}F_2. \tag{IV}$$

From the relation (IV) follows that

$$S - 4\sqrt{3}F \geq 0,$$

and that is the inequality of R Weitzenboeck[5]. For comparison, some other proofs and analogous inequalities can be found in T. Kubota[6].

Given a, b, c the side lengths of the triangle D, we denote

$$Q = (a-b)^2 + (b-c)^2 + (c-a)^2$$

and we have

$$8\sqrt{3}F_2 = 2S_2 \geq Q. \tag{V}$$

Together with (IV) we get the following tightening of Weitzenboeck's inequality:

$$S - Q - 4\sqrt{3}F \geq 0,$$

and denoting

$$u = a + b + c = \sqrt{3S - Q}$$

for the triangle D it follows that

$$u^2 - 2Q - 12\sqrt{3}F \geq 0.$$

Here, as in (V), the equality holds not only for a, b, c equal, but also when one side has the length zero.

If no angle in the original triangle D is greater than $2\pi/3$, and m is the existing minimum of the sum of the distances from a point in the plane from the three vertices of D then

$$m = \sqrt{4\sqrt{3}F_1} = \sqrt{S_1}. \tag{VI}$$

Since by (I) we have $F_1 = F + F_2$, from (VI) we deduce an estimate of U.T. Boedewadt[7],

$$m \geq \sqrt{4\sqrt{3}F}$$

[5] Math Zeitschnft 5 p. 137—146 (1919)

[6] Tohoku Math J 25, p. 122—126 (1925)

[7] Jahresbencht der D M V. 46 (1936), Losung der Aufgabe Nr. 196.

from which then again a weaker estimate by M. Schreiber[8] follows, namely

$$m \geq 6r,$$

where r denotes the radius of the incircle inscribed in D.

From (IV), (II) and (V) follows

$$m \leq \sqrt{ab + bc + ca}.$$

Let N be the minimum of the sum of squares of the distances of a point the plane from the three sides of the original triangle D. Then we have

$$N = \frac{F^2}{\sqrt{3}(F_1 + F_2)}. \tag{VII}$$

If the area of the largest equilateral triangle circumscribed to the original triangle D having the same or opposite orientation[9] is denoted by U_1 or U_2, then the following relations hold

$$U_1 = 4F_1, \quad U_2 = 4F_2. \tag{VIII}$$

If the areas of the smallest inscribed equilateral triangle in the original triangle D having the same or opposite orientation[10] are denoted by J_1 or J_2, then

$$J_1 = \frac{F^2}{4F_1}, \quad J_2 = \frac{F^2}{4F_2}. \tag{IX}$$

From (VIII) and (IX) we can infer that

$$F = \sqrt{U_1 J_1}, \quad F = \sqrt{U_2 J_2}. \tag{X}$$

The area of a triangle is equal to the geometric mean of the areas the largest circumscribed equilateral triangle and the smallest inscribed equilateral triangle.

3. We present now the proofs. If the angles assigned to the sides a, b, c are denoted by α, β and γ and the sides of the triangles D_1 and D_2 are labeled s_1 and s_2, we find easily by applying the law of cosines that

$$s_1 = \frac{a^2 + b^2 - 2ab\cos(\gamma + \frac{\pi}{3})}{3}$$
$$s_2 = \frac{a^2 + b^2 - 2ab\cos(\gamma - \frac{\pi}{3})}{3}.$$

We also have

$$2ab\cos\gamma = a^2 + b^2 - c^2$$

[8] Jahresbencht der D M. V. 45 (1935), Aufgabe Nr. 196.

[9] The corners of a circumtriangle are assigned to the corners of the original triangle in such a way that they do not have corresponding corners on a triangle side. The assigned corners can now result in the same or opposite sense of orientation.

[10] The original triangle is a circumscribed to this triangle. Then what is mentioned in Footnote 9) applies

$$ab\sin\gamma = 2F$$

$$a^2 + b^2 + c^2 = S$$

and plugging this into the previous expressions we get

$$s_1^2 = \frac{S + 4\sqrt{3}F}{6}$$

$$s_2^2 = \frac{S - 4\sqrt{3}F}{6}.$$

In the same way we obtain

$$S_1 = \frac{S + 4\sqrt{3}F}{2} \quad F_1 = \frac{\sqrt{3}S + 12F}{24}$$

$$S_2 = \frac{S - 4\sqrt{3}F}{2} \quad F_2 = \frac{\sqrt{3}S - 12F}{24},$$

from which the relations (I) and (II) as well as (III) and (IV) can be deduced.

If f is the area of the triangle with edge lengths $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ then

$$4f^2 \geq \sqrt{3}F.$$

Using the well known formula

$$16F^2 = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4)$$

we find after a few computations

$$4(16f^4 - 3F^2) = a^2(a-b)(a-c) + b^2(b-c)(b-a) + c^2(c-a)(c-b).$$

Assuming $a > b > c$, the first and third terms on the right hand side are positive, the second is negative, but of smaller magnitude than the first, so the whole expression is positive. It follows that

$$16f^2 = S - Q$$

and the above inequality together with (IV) leads to (V).

If the condition of (VI) is fulfilled regarding the angles of the original triangle D being smaller than $2\pi/3$, then the common point of intersection T of the three circumcircles of the equilateral triangles placed outwards lies in the interior of D. It is the so-called Torricelli point, which minimizes the sum of the distances to the three to the original corners. As radii of the circles mentioned above the following distances are equal:

$$TA_1 = CA_1, TB_1 = CB_1, \text{etc.}$$

From this it follows that the sides A_1B_1, etc. of the triangle D_1 and the segments TC, etc. are orthogonal and also halve them. Consequently the sum

$$m = TC + TB + TA,$$

representing the minimum in question, is equal to twice the sum of the distances of the point T from the three sides of the triangle D_1, and as such is equal to twice the height of D_1. This leads straight to the formula (VI).

The sum of squares of the distances from a point to the sides of triangle D is minimal for Lemoine's point L. Its distances from the sides are proportional to these. We can denote them by λa, λb, λc. The following equation obviously holds

$$\lambda a^2 + \lambda b^2 + \lambda c^2 = 2F,$$

so that the formula for the factor λ can be obtained from this

$$\lambda = \frac{2F}{S},$$

and so for the minimum in question is

$$N = \lambda^2 S = \frac{4F^2}{S} = \frac{F^2}{\sqrt{3}(F_1 + F_2)},$$

where (III) and (IV) were also taken into account.

Drawing straight lines through the vertices A, B, C, parallel to the sides B_1C_1, C_1A_1, A_1B_1, an equilateral triangle arises, having the same orientation with D, the area of which is $4 : F_1$, as can readily be seen from the evidence pertaining to (VI); the restricting assumption regarding (VI) is no longer necessary. This triangle is the largest circumscribed equilateral triangle as already shown in E. Fasbender[3] and can easily be confirmed as follows: The lines TA, TB, TC are normal to the sides of the triangle under consideration, as was already established earlier. If the three sides of the equilateral triangle rotated through the same angle, an equilateral triangle with the same orientation arises, but for that the sum of the three distances from T to the corresponding sides, becomes smaller. Since the total sum of these distances for an interior point of an equilateral triangle is equal to the height, the rotated triangle must be smaller.

For oppositely oriented triangles, the result given in (VIII) can be obtained in an analogous manner.

To prove (IX) pass through the vertices $A_1B_1C_1$ of D_1 straight lines parallel to BC, CA, AB. In this way one obtains a triangle D^*, that circumscribes D_1 and is similar to D. Since the distances of the parallel sides of the triangles D and D^* after the original construction of D_1 are proportional to the side lengths, D and D^* are in perspective and the center of similarity is the common Lemoine point L.

We further note that the vertices of the triangle D_1 are on the three mediatrix of the original triangle D, so that D_1 is the base triangle in D^* belonging to the circumcentre of D. As such is it the smallest equilateral equilateral intriangle of D^*[11]. If J_1 denotes the area of the smallest equilateral equilateral intriangle of D, then the following proportion applies:

$$\sqrt{J_1} : \sqrt{F_1} = a : a^*,$$

where a and a^* are the lengths of corresponding sides in the triangles D and D^*.

[11] Enzyklopadie der Math. Wiss. III AB 10, p. 1228.

If $\lambda = \frac{2F}{S}$ is the proportionality factor introduced in the proof of (VII), then the distances of the triangle sides mentioned above are from common Lemoine point L λa and λa^*. According to the original design, however

$$\lambda a^* = \lambda a + \frac{a}{2\sqrt{3}},$$

therefore we have

$$a : a^* = 1 : (1 + \frac{1}{2\sqrt{3}\lambda}),$$

or after inserting the value of λ given above

$$\sqrt{J_1} : \sqrt{F_1} = 4\sqrt{3}F : (S + 4\sqrt{3}F),$$

and using (III)

$$\sqrt{J_1} : \sqrt{F_1} = F : 2F_1,$$

from which

$$J_1 = \frac{F^2}{4F_1}$$

finishing the proof.

The case of oppositely oriented equilateral intriangles is handled in an analogue manner.

4. The relation (I) also holds if the nondegenerate attached triangles are only similar to each other and are positioned in such a way that at each corner of the original triangle corresponding (and therefore equal) angles arrive. The circumcircles of these triangles meet, depending on the attachment triangles with the original triangle on the opposite or on the same side of the common base, in a point T_1 or T_2[12] and their circumcenters form triangles D_1 or D_2, which are similar to the constructed triangles, since the angles match accordingly. For example for the vertex B in Fig. 1 we have $\angle B_1A_1C_1 = \frac{1}{2}\angle CA_1B$, equal to the peripheral angle over the chord BC. This shows that for the areas F, F_1, F_2 of D, D_1, D_2 the following relation is applicable

$$F_1 - F_2 = F.$$

Since the attachment triangles should not all degenerate and D_2 is similar to them, F_2 can only vanish if D_2 is reduced to a point. D_2 is then the common center of the three circles that must therefore coincide with the circumcircle of triangle D. In this case, the attached triangles that are set inwards must be congruent and therefore coincide with D, hence D_1 must be similar to D. In all other cases, $F_2 > 0$.

[12]Enzyklopadie der Math. Wiss. III AB 10, p. 1217.

If the point T_1 is reflected across the sides of D_1, one obtains the vertices A, B, C of D. By examining the corresponding sub-triangles, one sees that the area J of the polygon $AC_1BA_1CB_1$ equals $2F_1$. But on the other hand

$$J = F + \lambda a^2 + \mu b^2 + \nu c^2,$$

where the numbers λ, μ, ν depend only on the shape of the constructed triangles. Therefore we have

$$2F_1 = F + \lambda a^2 + \mu b^2 + \nu c^2.$$

Subtraction gives the equation we are looking for.

Special assumptions about the angles of the attachment triangles lead to the sentences:

Constructing over two sides of a triangle facing outwards or inwards equilateral triangles, the apex of one with the center of the other and the midpoint of the third side of the triangle form a triangle having angles 30°, 60°, 90°. *The difference in the areas of these triangles is equal to the area of the given triangle*

Constructing over two sides of a triangle as base outwards or inwards right-angled isosceles triangles, their vertices with the middle of the third side of the triangle form a right-angled isosceles triangle. The difference in the areas of these triangles is equal to the area of the given triangle.

It follows from this:

If two squares have a corner in common, then their centers form the opposite corners of a square whose other opposite corners are the midpoints of segments between corresponding corners of the given squares. The corresponding corners are adjacent to the common corner, but with opposite orientation.

If F_2 is 0 as above, then $F_1 = F$ and D_1 becomes congruent to D. So if you reflect the circumcenter of a triangle to the three sides, you get a congruent triangle.

More generally, a point P is called the mirror point of a nondegenerate triangle D if its mirror images with respect to the triangle sides a result in a triangle similar to D. These mirror points can be in determined in the following way:

The triangle D_1 is similar to D if and only if the top triangles are similar to D. The point T_1 is then the mirror point of D_1, and a similar mapping that transforms D_1 into D transforms T_1 into a mirror point of D. The same applies to D_2 and T_2, except in the case that D_2 reduces to one point.

To a certain side of a scalene triangle D one can put on a triangle similar to D but not coincident with D in 11 ways. This gives you all the mirror points, since the construction can also be reversed. Taking into account $F_1 - F_2 = F$ one finds:

A scalene triangle has 11 mirror points, one of which falls in the circumcenter and supplies a triangle congruent in the same direction. The others yield 5 equisimilar and 5 dissimilar triangles; at least 2 of the former and 3 of the latter are smaller than the given triangle. An isosceles triangle has 5 mirror points, at least two of these, smaller ones, and at least one[13]*yield a congruent triangle. In an equilateral triangle, the center of gravity is the only mirror point.*

In an analogous way, the points P can be determined, whose mirror images with respect to the given sides of a triangle, form one triangle with another predetermined shape. You get 12, or 6 or 2 such points, the more the resulting triangles

[13] For $b = c$ and $b : a = \sqrt{3 \pm \sqrt{7}}$ one obtains three triangles congruent to D.

become unequal. At least half of these are isosceles or equilateral triangles have a smaller area than the given one.

The mutual position of the points P is in connection with the base triangles, which are similar to the mirror image triangles, were already examined in [14]

5. In the general case considered in Section 4 the following applies:

The triangles D_1 and D_2 have the same orthocenter.

To show this, we may use the following theorem:

If a triangle changes similarly such that one corner stays fixed and a second corner runs on a straight line, then the third corner also runs on a straight line.

The triangle lies in a Gaussian number plane, its corners are determined by the complex numbers z_1, z_2, z_3. If $z_1 = 0$ is the fixed corner, then $z_3 = \text{const.}\ z_2$. If z_2 changes linearly, the same applies to z_3.

Due to the shape and arrangement of the attached triangles to D, the shape of the triangle CB_1A is determined. If the side BC of the triangle D is fixed and A is moved on a straight line g, then B_1 moves on a line. Since the triangle D_1 changes similarly and A_1 remains fixed, the orthocenter H_1 of D_1 also remains on a straight line g_1. If g is chosen perpendicular to BC, then g_1 is perpendicular to BC, because when A goes to infinity on g, the small angle between BC and C_1B_1 becomes arbitrarily small. If H_1 coincides with A_1 then H_1 remains fixed.

Now let A move continuously on g from the initial position to the mirror image $\overline{A}$ with respect to BC and then by reflection brought back to the initial position. Then D_1 goes over D_2 and H_1 becomes the orthocenter H_2 of D_2.

The points H_1 and H_2 are therefore at the same distance on the same side of g and also of the other heights of D, so they have to coincide.

In particular, if the vertices of D fall in a straight line, then we have $H_1 = H_2$, belonging to the same line. This gives the following result:

If three circles go through a point and the three other intersection points lie on a straight line, then the orthocenter of the triangle made by the circle centers also lies on this straight line.

The midpoints of the segments A_1A_2, B_1B_2 and C_1C_2 bisect the sides BC, CA and AB. So if you put the same masses in A, B, C and then distribute them half from B and C to A_1 and A_2, etc., then the center of gravity is not changed. The center of gravity of the triangle D is therefore the midpoint of the segment between the centroids of D_1 and D_2.

Because we have $H_1 = H_2$ the following result follows:

If the triangles D_1 and D_2 are equilateral, then their centroids coincide with the the centroid of D.

This theorem[15] also follows directly if one considers the triangles located in the Gaussian number plane. The corners of D are represented by the complex numbers z_1, z_2, z_3 and represent D_1 or D_2 by y_1, y_2, y_3. Then

$$y_1 = \frac{1}{2}(z_1 + z_2) \pm \frac{i}{2\sqrt{3}}(z_3 - z_2), \text{ and the similar relations.}$$

It follows that

$$\frac{1}{3}(y_1 + y_2 + y_3) = \frac{1}{3}(z_1 + z_2 + z_3).$$

[14] Enzyklopadie III AB 10, p. 1229.

[15] Which can be found in J. Neuberg, *Bibliographie du triangle et du tétraèdre* p 60; Mathésis 37 (1923), p. 452.

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