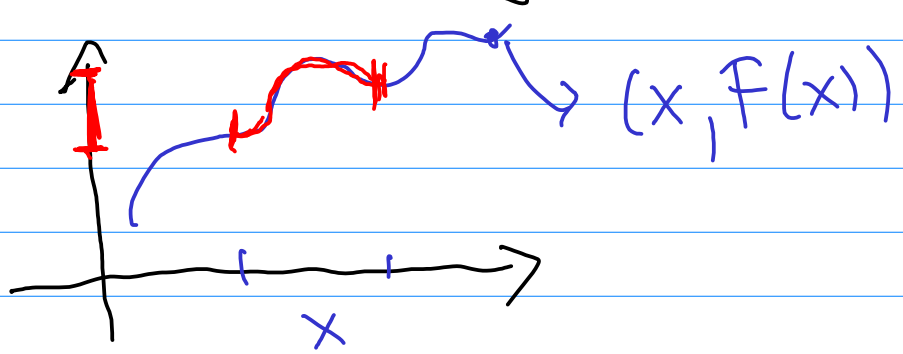
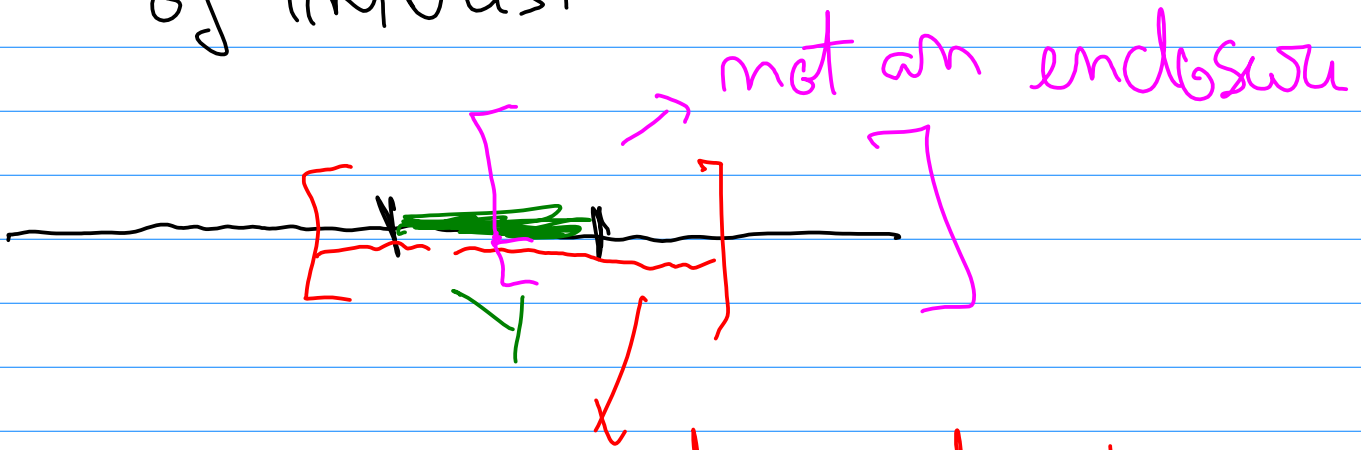
Enclosing a function image

X interval

estimate rigorously $F(X)$ where $F : \mathbb{R} \to \mathbb{R}$ is a function.

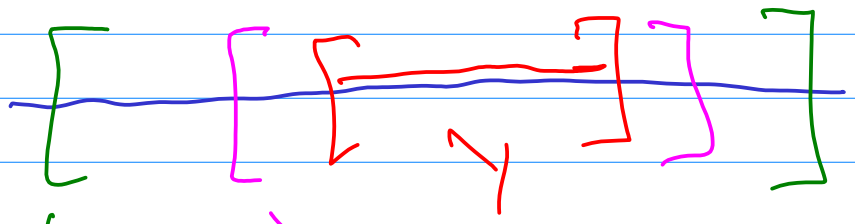


Enclosing: finding a BIGGER interval containing all objects of interest



Goal in Validated computing: find enclosure numerically which

is as tight/small as possible.



enclosure better enclosure.

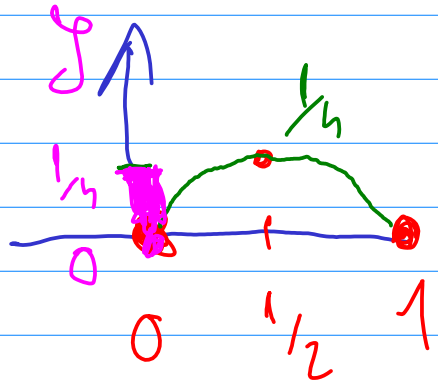
→ in practice enclosures are not exact

→ if we are careless: useless results

→ Assume basic interval operations could ok (addition, −, *, /)

Enclose $f(x) = x \cdot (1-x)$ on $[0,1]$

Math:



$f([0,1]) = [0, \frac{1}{4}]$

3 ways of computing (with intervals)

i) $f(x) = x - x^2 \quad \in [0,1] - [0,1] = [-1,1]$

$\in [0,1] \quad \in [0,1]$

Enclosure? YES $[0, \frac{1}{4}] \subset [-1,1]$

Not a tight enclosure.

ii) $f(x) = x(1-x) \quad \in [0,1] * [0,1] = [0,1]$

$\in [0,1] \quad \in [0,1]$

Enclosure? YES $[0, \frac{1}{4}] \subset [-1,1]$

Not a tight one

iii) $f(x) = \frac{1}{4} - \left(x - \frac{1}{2}\right)^2 \in \frac{1}{4} - [0, \frac{1}{4}]$

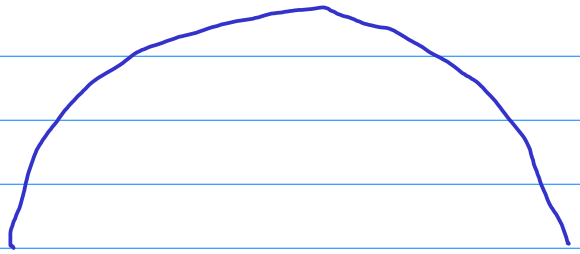
$\downarrow$ $\quad x - \frac{1}{2} \in [-\frac{1}{2}, \frac{1}{2}]$

$\frac{1}{4}$

$\left(x - \frac{1}{2}\right)^2 \in [0, \frac{1}{4}]$

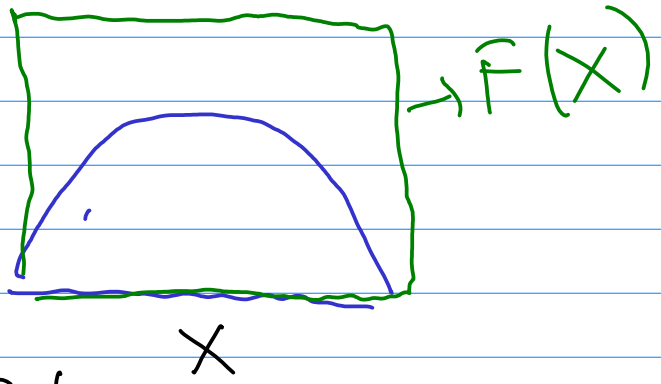
$\rightarrow f(x) \in [0, \frac{1}{4}] \rightarrow$ best enclosure

A general enclosure procedure.

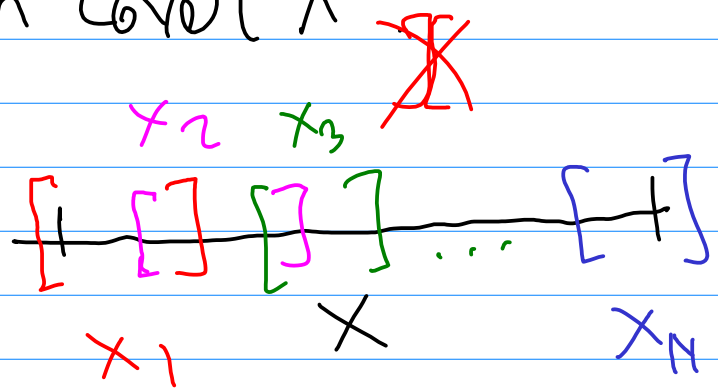


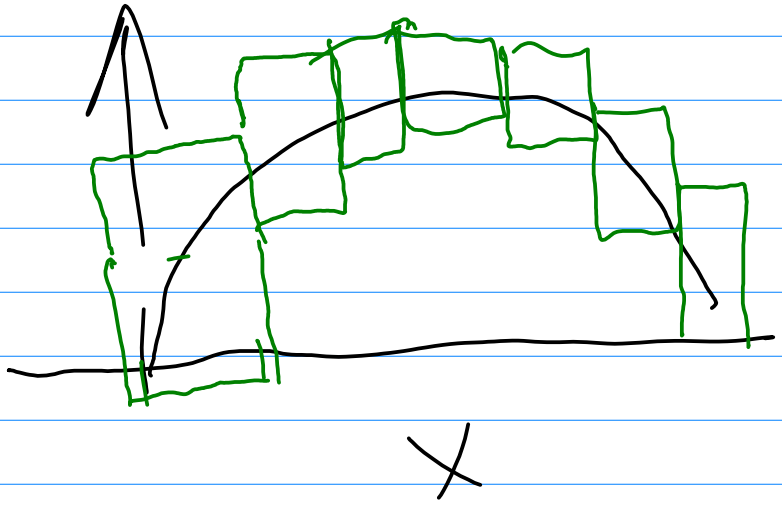
$F \rightarrow$ programmed function which encloses f

$F(X) \supseteq f(X)$ for every X



Split X into intervals $X_1, X_2 \dots X_N$ which cover X





X

$$f(X) \subseteq \text{hull}\{F(X_1), F(X_2) \dots F(X_n)\}$$

Sometimes: tighter than $F(X)$

Example: $f(X) = X \cdot (1 - X)$ $\quad 0 \le a < b \le 1$

$$F([a,b]) = [a,b] \cdot [1-b, 1-a]$$

$$= [a(1-b), b(1-a)]$$

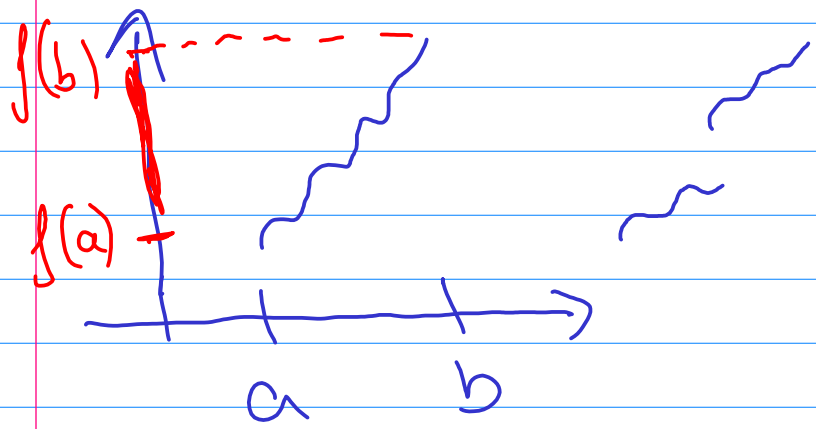
enclosure, not tightest

Numerically: using expression of f for eval enclosures is not enough.

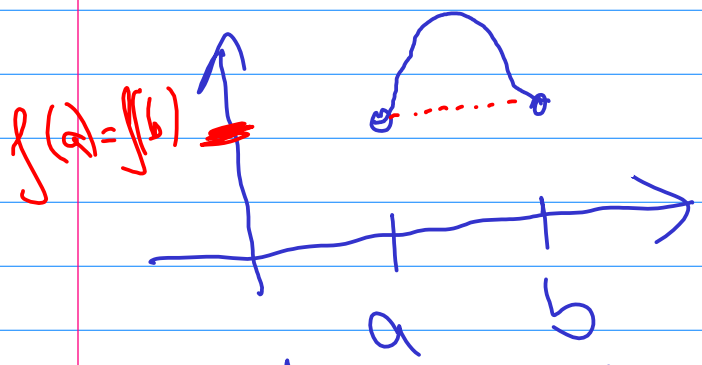
$$f([a,b]) = [f(a), f(b)]$$

Not ok in general

works only for continuous and increasing functions



discontinuous



An enclosure function (program) not the same as the function enclosed.

Certificate of exclusion:

a given value

Objective: $a \notin f(X)$

Applications: some threshold/value is never reached by the system.

System parameters X

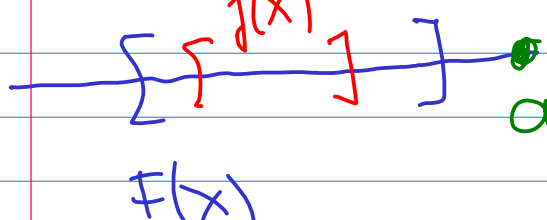
Check if $a \in \notin f(X)$?

If F computes enclosures for f and

$a \notin F(X) \Rightarrow a \notin f(X)$

Proof: $f(X) \subseteq F(X)$

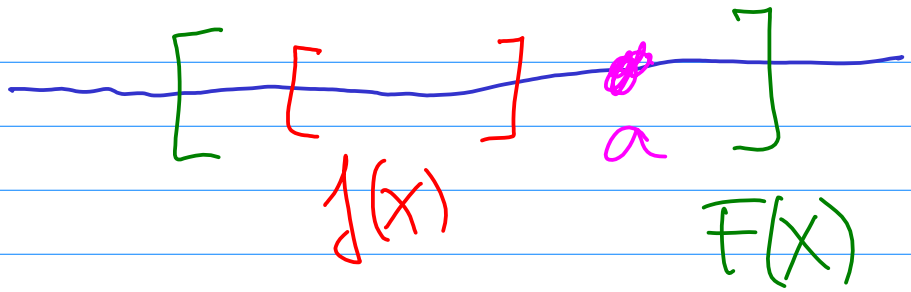
$a \notin F(X) \Rightarrow a \notin f(X)$



Converse is false:

if $F(X)$ contains a then we CANNOT conclude that $f(X)$ contains a.

Proof:



What we can prove with enclosures?

- prove that $f > 0$ on X?

  YES $F(X) \subset (0, \infty)$ ✓

- prove that $f(x) = 0$ has a solution? NO $0 \in F(X) \not\Rightarrow 0 \in f(X)$

$[\quad | \quad]$

depth 1 $[\quad][\quad]$ split 1/2

depth 2 $[I_1][I_2][I_3][I_4]$ split 1/2

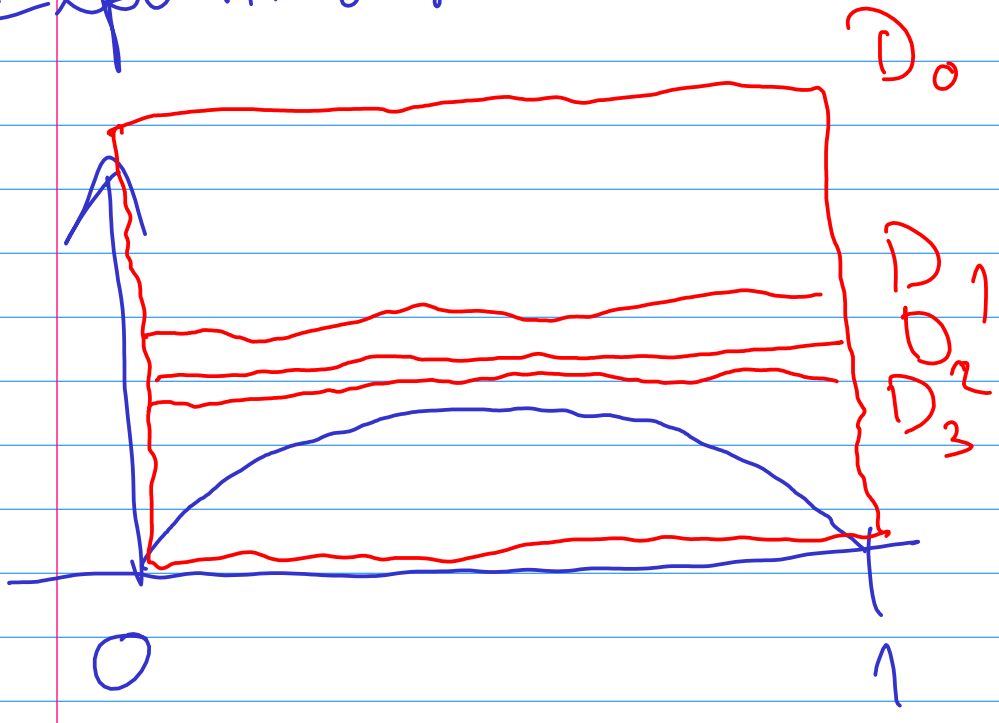
depth 3 $[\,][\,][\,][\,][\,][\,][\,][\,]$ split 1/2

Compute $F(I_1) = R_1$

Compute $R_2 = \text{hull}(R_1, F(I_2))$

...

Experiment



Experiment: expanded $f(x) = x - x^2$

