Optimal partitioning problems

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Optimal partitioning

Problem: Find a partition of a domain D which optimizes a certain quantity. Usually, the number of components of the partition is fixed, but there are problems where this number is unknown.

Examples: The hexagonal bee hive structure has the least cost in terms of the length of walls. This is a basic example of an optimal partitioning problem.

Difficulties: It is difficult to represent the sets of a partition in a general way. Any attempt in parametrizing the boundaries leads to troubles when dealing with the behavior of triple points.

Density representation: Instead of searching to parametrize the boundaries, we could *relax* the problem in the following way: we consider a family of functions with values in $[0,1]$ having their sum equal to 1.

1. Anisotropy

The shortest path from point A to point B is a straight line, but if between A and B there is a steep hill, then it might be shorter to go around. The anisotropic perimeter quantifies this aspect: a direction may be more costly than another.

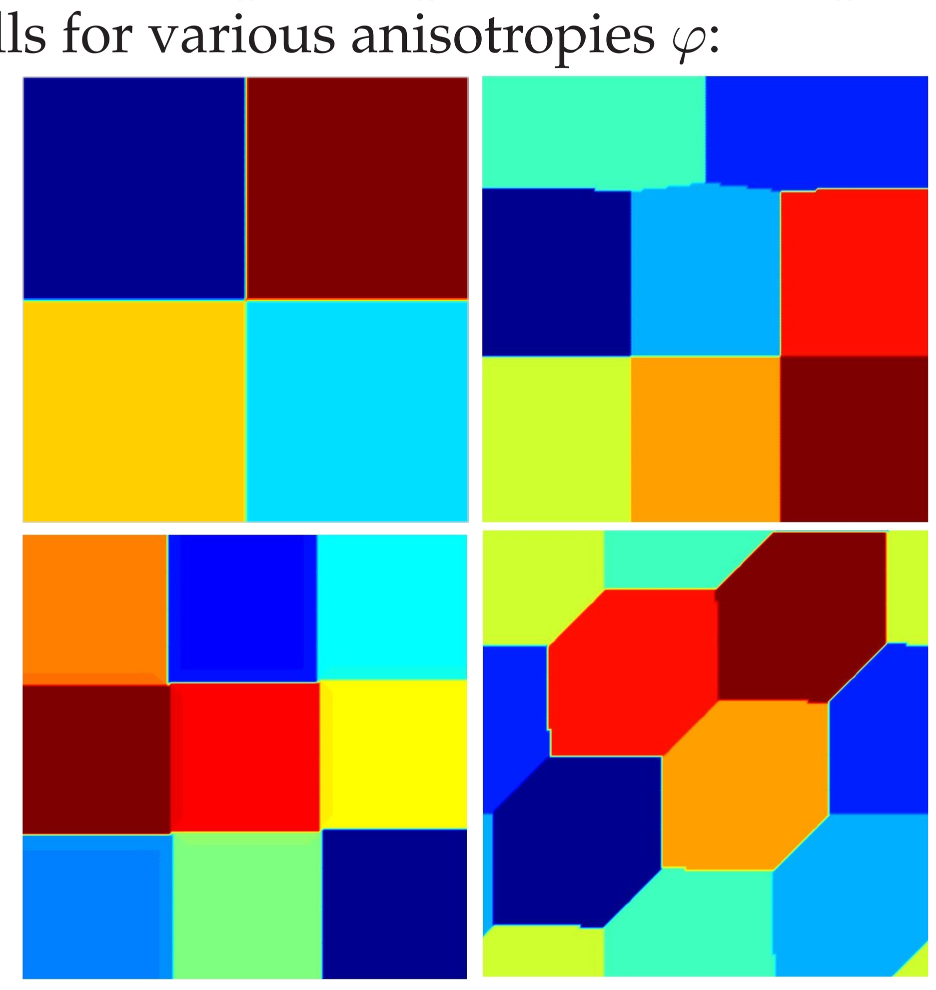
The isoperimetric problem says that the disk minimizes the perimeter at fixed area. If instead of the classical perimeter we consider an anisotropic perimeter which favorizes horizontal and vertical directions, then the shape which minimizes this anisotropic perimeter at fixed area is a square.

The anisotropic perimeter can be approximated using the following functional,

$$\text{Per}_\varphi(u) = \varepsilon \int_D \varphi(\nabla u)^2 + \frac{1}{\varepsilon}\int_D u^2(1-u)^2,$$

inspired by the Modica-Mortola theorem.

Using this representation, we can find numerical optimal partitions into equal area cells for various anisotropies φ:



Numerical results: The first three pictures $\varphi(x) = |x_1| + |x_2|$, in the last picture, a three directional anisotropy was considered.

2. Minimal perimeter partitions on the sphere

Problem: Find partitions of the sphere into N regions of equal areas, which minimize the total perimeter.

Interest: Despite of the fact that the problem is easy to state, theoretical results are known only for $N \in \{2,3,4,12\}$.

Previous works: Cox and Fikkema did numerical computations for $N \leq 32$, using the software Evolver.

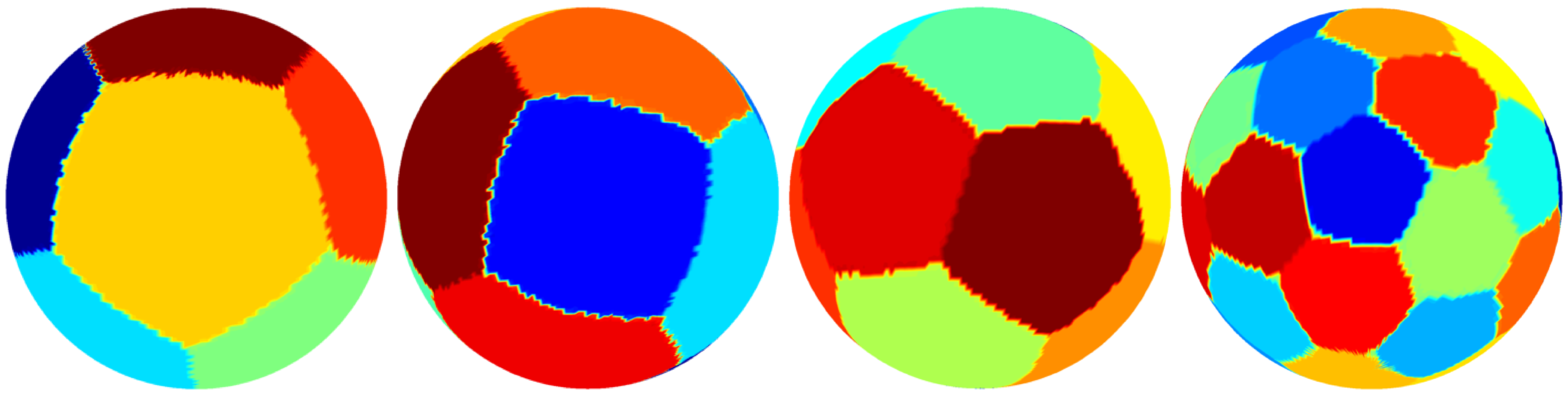
Our method: We approximate the spherical perimeter with the functional

$$F_\varepsilon(u) = \varepsilon \int_{\mathbb{S}^2} |\nabla_\tau u|^2 + \frac{1}{\varepsilon}\int_{\mathbb{S}^2} u^2(1-u)^2.$$

Numerical approach: We construct a uniform triangulation of the sphere, and use the rigidity and stiffness matrices given by the $P1$ finite elements associated to this triangilation in order to compute $F_\varepsilon(u_i)$ for every component of the partition. Then we preform a gradient descent algorithm.

Advantages: 1. The starting point is random; no initial assumption is made on the topological structure of the partition.

2. The method is really fast. It takes less than 5 minutes to compute numerically the optimal partition for $N = 32$.



Numerical results for $N \in \{7, 10, 12, 32\}$

3. Spectral minimal partitions on the sphere

Problem: Find partitions of the sphere into N regions which minimize the sum of their Laplace-Beltrami eigenvalues.

Interest: Little is known for these optimal partitions for $N \geq 3$. For $N = 3$ Bishop conjectured that the optimal partition consists of 3 lens of angle $2\pi/3$.

Previous works: Elliott and Ranner performed numerical computations using finite elements on surfaces for $N \in \{3,4,5,6,7,8,16,32\}$

Our method: We compute the eigenvalue of $\omega \subset \mathbb{S}^2$ by solving the penalized problem

$$-\Delta_\tau u + \mu u = \lambda u \text{ on } \mathbb{S}^2,$$

where $\mu >> 1$ in the complement of ω.

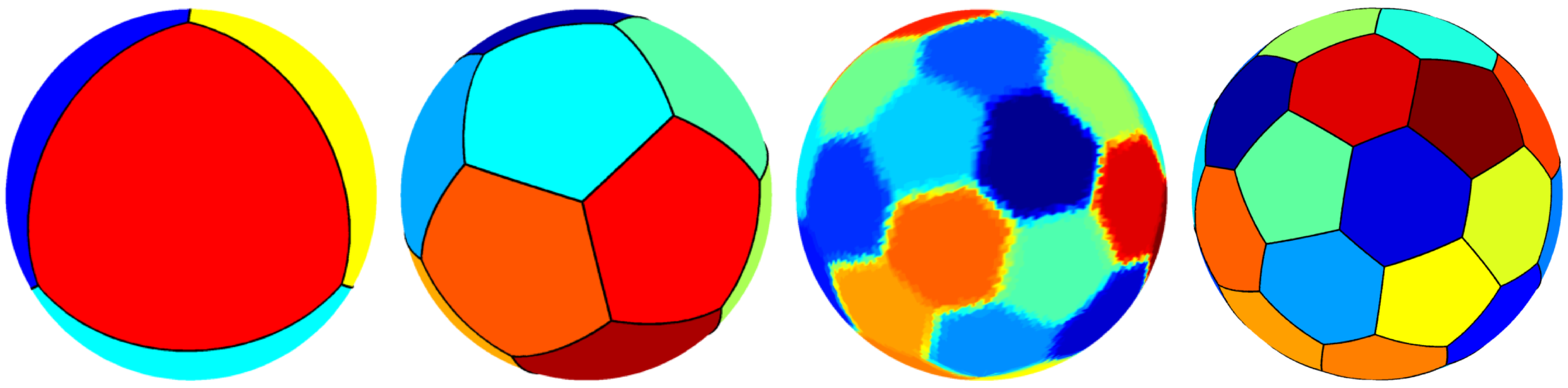
We discretize the problem using a method based on fundamental solutions:

- Choose (x_i) a family of points on $\mathbb{S}^2$ and (y_i) a family of centers outside $\mathbb{S}^2$.
- Consider the harmonic functions $\phi_i(x) = 1/|x - y_i|$
- Search for a solution in the form $u = \alpha_1\phi_1 + ... + \alpha_n\phi_n$. Note that for harmonic functions ϕ we have

$$-\Delta_\tau \phi = \frac{\partial \phi}{\partial n} + \frac{\partial^2 \phi}{\partial n^2}.$$

Numerical approach: We consider (x_i) at the vertices of a uniform triangulation of the sphere, and (y_i) on each normal corresponding to (x_i). We compute the eigenvalues using the procedure presented above. We impose the partition condition that the sum of all density functions is equal to one.

Advantages: The method of fundamental tolutions offers great computation accuracy. We were able to perform numerical computations for all $N \in \{3,4,5,...,23,24,32\}$. Visit my website (link below) to see all the results.



Numerical results (left to right): $N = 4$: regular tetrahedron partition, $N = 12$: regular pentagons tesselation, $N = 32$: density method and refined method. For $N = 32$ we find the C_{60} fullerene structure (like a soccer ball: 12 regular pentagons and 20 equal hexagons)

More details plus other topics

http://www.lama.univ-savoie.fr/~bogosel/